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Hyperplane sections of convex bodies

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Abstract

We prove sharp inequalities for the volumes of hyperplane sections bisecting a convex body in $\mathbb{R}^n$. This leads to a relative isoperimetric inequality for arbitrary hyperplane sections of a convex body.

Keywords: Hyperplane section; Relative isoperimetric inequality; pth moment

1. Introduction

This work was originally motivated by the following problem: find a sharp upper bound for the minimum of the volumes of the two parts into which a convex body K in $\mathbb{R}^n$ is divided by a hyperplane H in terms of the area of the slide $K \cap H$.

It should be remarked that this type of inequality is a simpler version of the so-called *relative isoperimetric inequalities,* which, for every (sufficiently smooth) subset E of an open bounded subset $G \subset \mathbb{R}^n$, relate the Lebesgue measure either of E or $G \setminus E$ to an appropriate $(n-1)$-dimensional measure of the boundary $\partial E \cap G$. Relative isoperimetric inequalities have been studied in connection with different topics, e.g. differential operators [18,19], immersion theorems for Sobolev spaces [2,5–7], geometry of numbers [3,14,24], and randomized algorithms for approximating the volume of a convex body [4,8,13].

Assumptions on G ensuring that a relative isoperimetric inequality holds are known (for details and references see [5]) and, when G is convex, explicit constants are known, depending on geometric properties of G [3,6,8,16]. In the present paper, we restrict our considerations to cuts by hyperplanes, and we seek a sharp upper bound on slides by parallel hyperplanes. For a given hyperplane H, let H^+ and H^- denote the two closed half-spaces bounded by H. Further, here and henceforth $|A|$ will denote the volume of A relative to its affine hull. Our aim is to obtain a sharp upper bound for the ratio

$$\rho(K,H) = \frac{\min\{|K \cap H^+|, |K \cap H^-|\}}{|K \cap H|},$$

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over the set of parallel hyperplane sections of K, in terms of the pth moments of inertia of a convex body K (see Section 2 for all terminology).

The pth moments have been extensively studied by Milman and Pajor in [20] and play a fundamental role in estimating the volumes of sections through the centroid of K as well as those of maximal volume, as it was pointed out by Ball [1], Milman and Pajor [20] for centrally symmetric convex bodies, and Fradelizi [10] in the general case. See also [9,21] for applications to concentration of mass in convex bodies.

The idea is to show that the maximum of the ratio $\rho(K,H)$, over the set of parallel hyperplane sections of K, is attained when H bisects K, that is H cuts K into two parts with equal volumes. Thus the problem can be reformulated as a question concerning the volumes of hyperplane sections bisecting K. Let $\widetilde{H}_u$ denote the hyperplane orthogonal to a given direction u which bisects K. By combining sharp estimates for the areas of the slides of K through its centroid [10] with the optimal bounds on the ratio of the areas of the slides through the centroid of K and the parallel hyperplane sections bisecting K [11], it follows that

$$\frac{|K|}{2|K\cap \widetilde{H}_u|} \leqslant c\left(\frac{p+1}{|K|}\int_K \left|\langle x,u\rangle\right|^p\,dx\right)^{\frac{1}{p}},$$

where $p \geqslant 1$, and

$$c = \max_{x>0} 2/\left(1+e^{-x}+\sqrt{1-(1+x)e^{-x}}\right) = 1.0629955\ldots.$$

We improve this inequality by showing that the constant c can be dropped. To get rid of the constant c, we study direct estimates of the areas of the hyperplane sections bisecting K, by adapting to our case the method used by Fradelizi in [10]. More generally, we prove the following theorem.

Theorem 1. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $\phi:\mathbb{R}\to\mathbb{R}$ be an even convex function. Let $u \in S^{n-1}$ and $\alpha = |K|/(2|K\cap \widetilde{H}_u|)$, where $\widetilde{H}_u$ denotes the hyperplane orthogonal to u which bisects K. Then*

$$\frac{1}{2}\int_{-1}^{1} \phi(\alpha t)\,dt \leqslant \frac{1}{|K|}\int_K \phi\big(\langle x,u\rangle\big)\,dx. \tag{1}$$

The inequality is sharp: if ϕ is strictly convex, there is equality if and only if K is a cylinder in the direction u.

As a direct consequence of this theorem applied with the even function $\phi(t) = |t|^p$, $p \geqslant 1$, we obtain the following estimate.

Corollary 2. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $u \in S^{n-1}$, and $p \geqslant 1$. Denote by $\widetilde{H}_u$ the hyperplane orthogonal to u which bisects K. Then*

$$\frac{|K|}{2|K\cap \widetilde{H}_u|} \leqslant \left(\frac{p+1}{|K|}\int_K \left|\langle x,u\rangle\right|^p\,dx\right)^{\frac{1}{p}}.$$

Moreover, there is equality if and only if K is a cylinder in the direction u.

This leads to the following relative isoperimetric inequality.

Theorem 3. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin, and $p \geqslant 1$. Let H_u be a hyperplane orthogonal to $u \in S^{n-1}$, which cuts K into two parts, say $K\cap H_u^+$ and $K\cap H_u^-$. Then*

$$\frac{\min\{|K\cap H_u^+|,|K\cap H_u^-|\}}{|K\cap H_u|} \leqslant \left(\frac{p+1}{|K|}\int_K \left|\langle x,u\rangle\right|^p\,dx\right)^{\frac{1}{p}}. \tag{2}$$

Equality holds if K is a cylinder in the direction u.

When K is centrally symmetric and $p = 1$, inequality (2) was previously proved in [22].

Generally, the upper bound in the above inequality depends on the direction u of the dividing hyperplane. However, when K is in isotropic position and $p = 2$, it involves the isotropic constant L_K whose best general known bound, due to Klartag [17], provides the following estimate.

Corollary 4. *Let K be an isotropic convex body in $\mathbb{R}^n$, and let H be a hyperplane that divides K into two parts, say $K \cap H^+$ and $K \cap H^-$. Then, for some absolute constant C,*

$$\frac{\min\{|K \cap H^+|, |K \cap H^-|\}}{|H \cap K|} \leqslant \sqrt{3} L_K < C n^{\frac{1}{4}}.$$

These results are presented as follows. Section 2 contains some basic definitions and preliminary results. In Section 3 we prove Theorem 1. In Section 4 we prove that a hyperplane section bisecting K has largest ratio $\rho(K, H)$ among all the parallel hyperplane sections, and combine this with the results of Section 3 to obtain Theorem 3.

2. Preliminaries

As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n-space $\mathbb{R}^n$. The canonical scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. If $u \in S^{n-1}$, $u^\perp = \{x \in \mathbb{R}^n\colon \langle x, u \rangle = 0\}$ denotes the hyperplane orthogonal to u through the origin. Hyperplanes parallel to $u^\perp$ will be parametrized in the form

$$H_u(t) = \big\{x \in \mathbb{R}^n\colon \langle x, u \rangle = t\big\},$$

where $t \in \mathbb{R}$.

If $A \subset \mathbb{R}^n$, we denote by $\chi_A : \mathbb{R} \to \mathbb{R}$ the characteristic function of A (i.e. $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$). Further, $|A|$ will denote the volume of A relative to its affine hull.

If $f : \mathbb{R} \to \mathbb{R}$, its support is denoted by support(f) (support(f) equals the closure of $\{x \in \mathbb{R}\colon f(x) \neq 0\}$). A function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be $\frac{1}{k}$-*concave* if support(f) is convex and $f^{1/k}$ is concave on support(f).

We recall that if K is a convex body in $\mathbb{R}^n$, i.e. a compact convex set with non-empty interior, its *centroid* is $\frac{1}{|K|} \int_K x\, dx$.

Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. For every $p > 0$ and $u \in S^{n-1}$, the *pth moment of inertia* of K is defined by

$$\left(\frac{1}{|K|} \int_K \big|\langle x, u \rangle\big|^p \, dx \right)^{\frac{1}{p}}.$$

We also say that K is *isotropic* or that K is in isotropic position if $|K| = 1$, the centroid of K is at the origin, and there is a constant L_K such that

$$\int_K \big|\langle x, u \rangle\big|^2 \, dx = L_K^2,$$

for all $u \in S^{n-1}$. The constant L_K is called the *isotropic constant*. For further information see the survey of Giannopoulos [12].

In the proof of Theorem 1 we shall use the following lemma proved by Fradelizi in [10, Lemma 3].

Lemma 5. *Let $v : \mathbb{R} \to \mathbb{R}$ be an integrable compactly supported function such that $\int_{-\infty}^{+\infty} v(t)\, dt = 0$ and $\int_{-\infty}^{+\infty} t v(t)\, dt = 0$. Set $V(t) = \int_{-\infty}^{t} v(s)\, ds$. Let ϕ be a convex function and μ be the positive Borel measure on $\mathbb{R}$ such that $\phi'' = \mu$. Then the function $W(t) = \int_{-\infty}^{t} V(s)\, ds$ is compactly supported and*

$$\int_{-\infty}^{+\infty} \phi(t) v(t)\, dt = \int_{-\infty}^{+\infty} W(t)\, d\mu(t).$$

We also need the following result.

Lemma 6. *Let $b, c > 0$ and $\tilde{t} \in (-c, b)$ be such that*

$$\int_{-c}^{\tilde{t}} e^{t-\tilde{t}}\,dt = \int_{\tilde{t}}^{b} e^{t-\tilde{t}}\,dt, \qquad \int_{-c}^{b} t e^{t-\tilde{t}}\,dt = 0.$$

Then

$$\frac{4bc}{b+c} \geqslant \int_{-c}^{b} e^{t-\tilde{t}}\,dt.$$

Proof. Since $\int_{-c}^{b} t e^{t-\tilde{t}}\,dt = 0$, then $\int_{-c}^{b} t e^{t}\,dt = 0$ so that $be^b + ce^{-c} = e^b - e^{-c}$. If we set $x = \frac{b+c}{2}$, then we have $c = \frac{xe^x}{\sinh x} - 1$ and $b = 1 - \frac{xe^{-x}}{\sinh x}$. Since $\int_{-c}^{\tilde{t}} e^{t-\tilde{t}}\,dt = \frac{1}{2}\int_{-c}^{b} e^{t-\tilde{t}}\,dt$, we have $e^{\tilde{t}} = \frac{1}{2}(e^b + e^{-c})$, so that $\int_{-c}^{b} e^{t-\tilde{t}}\,dt = \frac{2(e^b - e^{-c})}{e^b + e^{-c}}$. Hence it is enough to prove that for all $x > 0$,

$$\left(\frac{xe^x}{\sinh x} - 1\right)\left(1 - \frac{xe^{-x}}{\sinh x}\right) - \frac{x \sinh x}{\cosh x} \geqslant 0.$$

We have

$$\left(\frac{xe^x}{\sinh x} - 1\right)\left(1 - \frac{xe^{-x}}{\sinh x}\right) - \frac{x \sinh x}{\cosh x} = \frac{(e^{4x} - 4xe^{2x} - 1)(e^{2x}(x-1) + x + 1)}{(e^{2x}+1)(e^{2x}-1)^2}.$$

Let us consider the functions $F(x) = e^{4x} - 4xe^{2x} - 1$ and $G(x) = e^{2x}(x-1) + x + 1$. Since $\lim_{x\to 0} F(x) = \lim_{x\to 0} G(x) = 0$, and for all $x > 0$ we have $F'(x) > 0$ and $G'(x) > 0$, we obtain the desired result. □

We recall that a function $f : \mathbb{R}^k \to [0, \infty)$ is *log-concave* if support(f) is convex and the function $\ln f : \mathbb{R}^k \to [-\infty, \infty)$ is concave (with the usual convention regarding $-\infty$). The following lemma for log-concave functions was proved by Kannan, Lovász and Simonovits in [16]. It should be remarked that this method was also introduced by Gromov and Milman in [15] to study the spherical isoperimetric inequality.

Lemma 7. *Let f_1, f_2, f_3, f_4 be four non-negative continuous functions defined on an interval $[a, b] \subset \mathbb{R}$, and let $\alpha, \beta > 0$. Then the following are equivalent*:

(i) *For every log-concave function F on $\mathbb{R}$,*

$$\left(\int_a^b F(t) f_1(t)\,dt\right)^{\alpha} \left(\int_a^b F(t) f_2(t)\,dt\right)^{\beta} \leqslant \left(\int_a^b F(t) f_3(t)\,dt\right)^{\alpha} \left(\int_a^b F(t) f_4(t)\,dt\right)^{\beta}.$$

(ii) *For every subinterval $[a', b'] \subset [a, b]$, and every real number ζ,*

$$\left(\int_{a'}^{b'} e^{\zeta t} f_1(t)\,dt\right)^{\alpha} \left(\int_{a'}^{b'} e^{\zeta t} f_2(t)\,dt\right)^{\beta} \leqslant \left(\int_{a'}^{b'} e^{\zeta t} f_3(t)\,dt\right)^{\alpha} \left(\int_{a'}^{b'} e^{\zeta t} f_4(t)\,dt\right)^{\beta}.$$

By dominated convergence of the integrals this lemma extends to the more general case when f_1 and f_2 are upper semicontinuous functions, and f_3 and f_4 are lower semicontinuous functions.

We also need the following lemma which is an immediate consequence of the Brunn–Minkowski theorem [23].

Lemma 8. *Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. The function $f(t) = |K \cap H_u(t)|$, for $t \in \mathbb{R}$, is log-concave.*

3. Proof of Theorem 1

We first give a functional version of Theorem 1. Let $f(t)=|K\cap H_u(t)|$ and let $\tilde{t}\in\mathbb{R}$ such that the hyperplane $\widetilde{H}_u := H_u(\tilde{t})$ bisects K. We can express the quantities in (1) in terms of f, as follows:

$$\alpha=\frac{\int_{-\infty}^{+\infty} f(t)\,dt}{2f(\tilde{t})}=\frac{\int_{-\infty}^{\tilde{t}} f(t)\,dt}{f(\tilde{t})}=\frac{\int_{\tilde{t}}^{+\infty} f(t)\,dt}{f(\tilde{t})},$$

$$|K|=\int_{-\infty}^{+\infty} f(t)\,dt \quad\text{and}\quad \int_K \phi\big(\langle x,u\rangle\big)\,dx=\int_{-\infty}^{+\infty}\phi(t)f(t)\,dt.$$

Since the centroid of K is at the origin, then we have

$$\int_{-\infty}^{+\infty} tf(t)\,dt=0,$$

where the function f is $\frac{1}{n-1}$-concave by the Brunn–Minkowski theorem [23]. To prove inequality (1) it is enough to prove the following inequality:

$$f(\tilde{t})\int_{-\alpha}^{\alpha}\phi(t)\,dt\leqslant\int_{-\infty}^{+\infty}\phi(t)f(t)\,dt \tag{3}$$

for every $\frac{1}{n-1}$-concave integrable function $f:\mathbb{R}\to\mathbb{R}^+$ such that $\int_{-\infty}^{+\infty} tf(t)\,dt=0$.

Since f is a $\frac{1}{n-1}$-concave integrable function, it is continuous in the interior of its support, which is a bounded interval. Hence, we may assume that f is continuous on its support, as we are interested in integration properties of f. We may also assume $f(\tilde{t})=1$.

We now prove inequality (3), by adapting to our case the method used by Fradelizi in [10].

The proof consists of two steps. We first reduce to the log-affine case (see Proposition 9 below), and we then show that among all log-affine functions the quantity $\int_{-\infty}^{+\infty}\phi(t)g(t)\,dt$ is minimal when g is constant on its support (see Proposition 10).

Proposition 9. *There exist $a,b,c\in\mathbb{R}$, where $b,c>0$ (depending on f and $\tilde{t}$), such that $[-c,b]\subset\operatorname{support}(f)$, and the function $g(t)=e^{a(t-\tilde{t})}\chi_{[-c,b]}(t)$ satisfies the conditions*

$$\int_{-\infty}^{\tilde{t}} g(t)\,dt=\int_{-\infty}^{\tilde{t}} f(t)\,dt,\qquad \int_{\tilde{t}}^{+\infty} g(t)\,dt=\int_{\tilde{t}}^{+\infty} f(t)\,dt,\qquad \int_{-\infty}^{+\infty} tg(t)\,dt=0, \tag{4}$$

and

$$\int_{-\infty}^{+\infty}\phi(t)g(t)\,dt\leqslant\int_{-\infty}^{+\infty}\phi(t)f(t)\,dt \tag{5}$$

for every convex function $\phi:\mathbb{R}\to\mathbb{R}$.

Proof. We first build a class of log-affine functions $g(t)=e^{a(t-\tilde{t})}\chi_{[-c,b]}(t)$ which satisfy the first two relations in (4). Let

$$a_2=\left(\int_{\tilde{t}}^{+\infty} f(t)\,dt\right)^{-1}=\left(\int_{-\infty}^{\tilde{t}} f(t)\,dt\right)^{-1}.$$

For $a \in [-a_2, a_2]$, we define $b(a) = \tilde{t} + \frac{1}{a}\ln(1 + \frac{a}{a_2})$ if $a \neq 0$, and $b(0) = \tilde{t} + \frac{1}{a_2}$, $c(a) = \tilde{t} + \frac{1}{a}\ln(1 - \frac{a}{a_2})$ if $a \neq 0$, and $c(0) = \tilde{t} - \frac{1}{a_2}$. Note that $c(a) \leqslant \tilde{t} \leqslant b(a)$, and the functions $b(a)$ and $c(a)$ are continuous and decreasing on $[-a_2, a_2]$. Define the function $g_a(t) = e^{a(t-\tilde{t})}\chi_{[c(a),b(a)]}(t)$. Then

$$\int_{\tilde{t}}^{+\infty} g_a(t)\,dt = \int_{\tilde{t}}^{+\infty} f(t)\,dt \quad \text{and} \quad \int_{-\infty}^{\tilde{t}} g_a(t)\,dt = \int_{-\infty}^{\tilde{t}} f(t)\,dt. \tag{6}$$

We now select a function within this class, which satisfies the last equality in (4). To this end, we consider the function $\Psi(a) = \int_{-\infty}^{+\infty} t(f(t) - g_a(t))\,dt$, and we show that there exists $a \in [-a_2, a_2]$ such that $\Psi(a) = 0$. Since Ψ is continuous on $[-a_2, a_2]$, it is enough to prove that $\Psi(-a_2) \leqslant 0$ and $\Psi(a_2) \geqslant 0$.

Let $F(x) = \int_{-\infty}^{x} f(t)\,dt$ and $G_a(x) = \int_{-\infty}^{x} g_a(t)\,dt$; the function $F - G_a$ is compactly supported for $a \in (-a_2, a_2)$. Hence

$$\Psi(a) = \int_{-\infty}^{+\infty} t\big(f(t) - g_a(t)\big)\,dt = -\int_{-\infty}^{+\infty} F(x) - G_a(x)\,dx.$$

Note that the previous equality holds true also for $a = -a_2$ and $a = a_2$.

Further, let us consider the function $\varphi(t) = (\ln f(t))/(t - \tilde{t})$, and let us define $a_1 = \lim_{t \to \tilde{t}^+} \varphi(t)$. We show that $|a_1| \leqslant a_2$. Since f is log-concave, the function φ is non-increasing on support(f) so that $f(t) \leqslant e^{a_1(t-\tilde{t})}$. If $a_1 < 0$, then $a_1 < a_2$ and

$$\frac{1}{a_2} = \int_{\tilde{t}}^{+\infty} f(t)\,dt \leqslant \int_{\tilde{t}}^{+\infty} e^{a_1(t-\tilde{t})}\,dt = -\frac{1}{a_1}$$

so that $a_1 \geqslant -a_2$. If $a_1 > 0$, then $a_1 > -a_2$ and

$$\frac{1}{a_2} = \int_{-\infty}^{\tilde{t}} f(t)\,dt \leqslant \int_{-\infty}^{\tilde{t}} e^{a_1(t-\tilde{t})}\,dt = \frac{1}{a_1}$$

so that $a_1 \leqslant a_2$.

We are now ready to prove that $\Psi(-a_2) \leqslant 0$. Since $b(-a_2) = +\infty$, for all $t > \tilde{t}$ the function $f(t) - g_{-a_2}(t) = e^{(t-\tilde{t})\varphi(t)} - e^{-a_2(t-\tilde{t})}$ has the same sign as $\varphi(t) - a_2$, which is non-increasing. By (6) $\int_{\tilde{t}}^{+\infty}(f(t) - g_{-a_2}(t))\,dt = 0$. Hence there exists $t_o > \tilde{t}$ such that the function $f - g_{-a_2}$ changes sign at t_o. Therefore

- on $[\tilde{t}, t_o]$, $\varphi(t) \geqslant a_2$, and $f \geqslant g_{-a_2}$, so that $F(x) - G_{-a_2}(x)$ is non-decreasing;
- on $[t_o, +\infty]$, $\varphi(t) \leqslant a_2$, and $f \leqslant g_{-a_2}$, so that $F(x) - G_{-a_2}(x)$ is non-increasing.

For all $t \leqslant \tilde{t}$, since $a_1 \geqslant -a_2$, we have $f(t) \leqslant e^{a_1(t-\tilde{t})} \leqslant e^{-a_2(t-\tilde{t})}$. Then

- on $[c(-a_2), \tilde{t}]$, $f \leqslant g_{-a_2}$, so that $F(x) - G_{-a_2}(x)$ is non-increasing;
- on $(-\infty, c(-a_2))$, $f - g_{-a_2} = f$, so that $F(x) - G_{-a_2}(x)$ is non-decreasing.

Furthermore, since $(F - G_{-a_2})(-\infty) = (F - G_{-a_2})(\tilde{t}) = (F - G_{-a_2})(+\infty) = 0$, we have $F - G_{-a_2} \geqslant 0$. Hence $\Psi(-a_2) \leqslant 0$. Moreover, $F - G_{-a_2} \geqslant 0$ and the continuity of $F - G_{-a_2}$ imply $\Psi(-a_2) < 0$.

Analogously we have $\Psi(a_2) > 0$. By the continuity of Ψ it follows that there exists $a_o \in (-a_2, a_2)$ such that $\Psi(a_o) = 0$. This prove that g_{a_o} satisfies equalities (4). For the sake of simplicity we denote $g := g_{a_o}$, $c := -c(a_o) > 0$ and $b := b(a_o) > 0$.

To prove that g also satisfies (5) we apply Lemma 5 to $v = f - g$. Hence it is enough to show that the "second primitive" $W(t)$ of $v = f - g$ is non-negative.

Since $\int_{\tilde{t}}^{+\infty} v(t)\,dt = \int_{-\infty}^{\tilde{t}} v(t)\,dt = 0$, the sign of v changes on $(\tilde{t}, +\infty)$ and on $(-\infty, \tilde{t})$. We consider two cases.

1) Assume that v keeps the same sign on $[-c,b] = \text{support}(g)$. Since outside $[-c,b]$ we have $v = f \geqslant 0$, then $v \leqslant 0$ on $[-c,b]$. Since W is compactly supported and $W'' = v$, we get $W \geqslant 0$ and $\text{support}(g) \subset \text{support}(f)$.

2) Assume that the sign of v changes at $t_o \in [-c,b]$. Without loss of generality, we may assume $t_o \geqslant \tilde{t}$. On $[\tilde{t}, b]$ the function $v(t) = e^{(t-\tilde{t})\varphi(t)} - e^{a_o(t-\tilde{t})}$ has the same sign as $\varphi(t) - a_o$, which is non-increasing. Hence we get $v \geqslant 0$ on $[\tilde{t}, t_o]$ and $v \leqslant 0$ on $[t_o, b]$. On $[-c, \tilde{t}]$ we have $v(t) \leqslant 0$ like $a_o - \varphi(t)$. Outside $[-c,b]$, we have $v = f \geqslant 0$. Since W is compactly supported and $W'' = v$, we get $W \geqslant 0$ and $\text{support}(g) \subset \text{support}(f)$. □

We now prove that among all log-affine functions g which satisfy (4) and (5) the quantity $\int_{-\infty}^{+\infty} \phi(t) g(t)\, dt$ is minimal when g is constant on its support.

Proposition 10. *Let $a, b, c, \tilde{t} \in \mathbb{R}$, where $b, c > 0$, be such that the function $g(t) = e^{a(t-\tilde{t})}\chi_{[-c,b]}(t)$ satisfies the equalities*:

$$\int_{-c}^{\tilde{t}} g(t)\, dt = \int_{\tilde{t}}^{b} g(t)\, dt, \qquad \int_{-c}^{b} t g(t)\, dt = 0. \tag{7}$$

Set $d := \frac{1}{2}\int_{-\infty}^{+\infty} g(t)\, dt$. Then, for every even convex function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\int_{-d}^{d} \phi(t)\, dt \leqslant \int_{-c}^{b} \phi(t) g(t)\, dt.$$

Proof. By a change of variable we reduce to the case when $a = 1$. For a given $\tilde{t} \in \mathbb{R}$, we set $x := (b+c)/2$, so that b and c are positive functions of $x > 0$. Further, the integrals in (7) and the value d, which depend on b and c, become $\mathcal{C}^\infty$ functions of x. By differentiating the equality $\int_{-c(x)}^{b(x)} t e^{(t-\tilde{t})}\, dt = 0$, valid for all $x > 0$, we get $b'(x) b(x) e^{(b(x)-\tilde{t})} = c'(x) c(x) e^{(-c(x)-\tilde{t})}$. Differentiating $d(x) = \frac{1}{2}\int_{-c(x)}^{b(x)} e^{(t-\tilde{t})}\, dt$, we obtain

$$d'(x) = \frac{1}{2}\bigl(b'(x) e^{(b(x)-\tilde{t})} + c'(x) e^{(-c(x)-\tilde{t})}\bigr) = \frac{1}{2} b'(x) e^{(b(x)-\tilde{t})}\left(\frac{b(x)+c(x)}{c(x)}\right).$$

Let us consider the function

$$I(x) := \int_{-c(x)}^{b(x)} \phi(t) e^{(t-\tilde{t})}\, dt - \int_{-d(x)}^{d(x)} \phi(t)\, dt.$$

Since $I(0) = 0$, it is enough to show that $I'(x) > 0$. Since ϕ is even we have

$$I' = b'\phi(b) e^{(b-\tilde{t})} + c'\phi(-c) e^{(-c-\tilde{t})} - 2d'\phi(d) = b' e^{(b-\tilde{t})}\left(\frac{b+c}{c}\right)\left(\frac{c}{b+c}\phi(b) + \frac{b}{b+c}\phi(c) - \phi(d)\right).$$

Since ϕ is convex, even, and $b' > 0$, from Lemma 6 we get

$$I' \geqslant b' e^{(b-\tilde{t})}\left(\frac{b+c}{c}\right)\left(\phi\left(\frac{2bc}{b+c}\right) - \phi(d)\right) \geqslant 0. \qquad \square$$

Inequality (3) then follows from Propositions 9 and 10.

3.1. The equality case

If ϕ is an even strictly convex function and μ is the positive Borel measure on $\mathbb{R}$ such that $\phi'' = \mu$, then $\text{support}(\mu) = \mathbb{R}$ and the equality case in (1) can be characterized. In fact if there is equality in (1), then there is equality in the corresponding functional form (3), so that there is equality in (5), where $g(t) = f(\tilde{t})\chi_{[-\alpha,\alpha]}(t)$. Then from Lemma 5 the second primitive W of $v = f - g$ satisfies $\int_{-\infty}^{+\infty} W(t)\, d\mu(t) = 0$. Since μ is positive and W is non-negative, then $f = g$ almost everywhere. Since f and g are continuous on their support, it follows that $f = g$. Thus the equality case stated in Theorem 1 follows from Brunn–Minkowski theorem.

4. Proof of Theorem 3

Let us now cut a convex body K with a family of parallel hyperplanes orthogonal to a given direction $u \in S^{n-1}$. For $t \in \mathbb{R}$ the hyperplane $H_u(t)$ bounds the two closed half-spaces

$$H_u^-(t) := \{x \in \mathbb{R}^n \colon \langle x, u \rangle \leqslant t\},$$
$$H_u^+(t) := \{x \in \mathbb{R}^n \colon \langle x, u \rangle \geqslant t\}.$$

Denote by $[a, b]$ the support of the function $f(t) := |K \cap H_u(t)|$. For $a < t < b$ we consider the function ψ defined by

$$\psi(t) := \frac{|K \cap H_u^-(t)|}{|K \cap H_u(t)|}.$$

Proposition 11. *Let $a < x < y < b$. Then $\psi(x) \leqslant \psi(y)$.*

Proof. It is enough to prove that for $h > 0$ sufficiently small

$$\int_a^x f(t)\,dt \int_y^{y+h} f(t)\,dt \leqslant \int_a^y f(t)\,dt \int_x^{x+h} f(t)\,dt, \tag{8}$$

as this implies the result by letting $h \to 0$.

Let h be sufficiently small so that $x + h < y < y + h < b$. Denote by f_1, f_2, f_3, f_4 the characteristic functions of the intervals $[a, x]$, $[y, y + h]$, $[a, y]$, $[x, x + h]$, respectively. Then, inequality (8) becomes

$$\int_a^b f(t) f_1(t)\,dt \int_a^b f(t) f_2(t)\,dt \leqslant \int_a^b f(t) f_3(t)\,dt \int_a^b f(t) f_4(t)\,dt.$$

By Lemmas 7 and 8 it is enough to prove that for every subinterval $[a', b'] \subset [a, b]$ and every real number ζ,

$$\int_{a'}^{b'} e^{\zeta t} f_1(t)\,dt \int_{a'}^{b'} e^{\zeta t} f_2(t)\,dt \leqslant \int_{a'}^{b'} e^{\zeta t} f_3(t)\,dt \int_{a'}^{b'} e^{\zeta t} f_4(t)\,dt.$$

We can assume $[x, y] \subset [a', b']$, otherwise the inequality is trivial. If $y < b' \leqslant y + h$, then

$$\int_{a'}^{b'} e^{\zeta t} f_2(t)\,dt \leqslant \int_y^{y+h} e^{\zeta t} f_2(t)\,dt,$$

and the result follows easily from the case $[x, y + h] \subset [a', b']$. Thus, we can assume $[x, y + h] \subset [a', b']$. We can also rescale so that $\zeta = 1$ (the case $\zeta = 0$ is trivial). Then, the inequality we have to prove becomes

$$\int_{a'}^x e^t\,dt \int_y^{y+h} e^t\,dt \leqslant \int_{a'}^y e^t\,dt \int_x^{x+h} e^t\,dt$$

which is trivially true for $a' \leqslant x < y + h \leqslant b'$. $\square$

Notice that Proposition 11 implies that the maximum of the ratio

$$\rho(K, u, t) = \frac{\min\{|K \cap H_u^-(t)|, |K \cap H_u^+(t)|\}}{|K \cap H_u(t)|},$$

on $t \in (a, b)$, is attained when $H_u(t)$ bisects K. In fact, let $\tilde{t} \in (a, b)$ be such that $|K \cap H_u^-(\tilde{t})| = |K \cap H_u^+(\tilde{t})|$. If $a < t \leqslant \tilde{t}$, we have $|K \cap H_u^-(t)| \leqslant |K \cap H_u^+(t)|$ so that

$$\rho(K, u, t) = \frac{|K \cap H_u^-(t)|}{|K \cap H_u(t)|} = \psi(t) \leqslant \psi(\tilde{t}) = \rho(K, u, \tilde{t}).$$

The case $\tilde{t} < t < b$ reduces to the previous one by a symmetry with respect to the hyperplane $H_u(\tilde{t})$.

Theorem 3 then follows from Proposition 11 and Corollary 2.

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